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Complete branching rules generating function for SO(7) ⊃ SU(2)³ and polynomial basis states*

Č Burdík†, C J Cummins‡, R W Gaskell§ and R T Sharp‖

† Nuclear Centre, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, CS-180 00 Prague 8, Czechoslovakia
‡ Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Boulevard West, H3G 1M8, Montreal, Quebec, Canada
§ Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109, USA
‖ Physics Department, McGill University, Montreal, Quebec, H3A 2T8, Canada

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Abstract. A branching rules generating function is given for $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$, along with its interpretation in terms of basis states, and instructions for calculating generator matrix elements. The generator matrix elements for the degenerate $(0,0,c)$ representations are calculated as an example of this procedure.

1. Introduction

The group–subgroup pair $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$ has been discussed by Van den Berghe *et al* (1982); they treat the symmetric representations $(a,0,0)$ in detail and give the relevant generator matrix elements. In a subsequent paper (Van der Jeugt and De Wilde 1984) the shift operator technique is developed for the problem in question; generator matrix elements for $(a,0,0)$, $(a,0,1)$ and $(0,0,c)$ are given.

In section 2 we use generating function methods to find the complete branching rules for $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$. In section 3 it is shown how the branching rule generating function can be used to define polynomial basis states (in the states of fundamental irreducible representations), and how the states can be used to derive generator matrix elements. The methods apply to any group–subgroup pair. In section 4 we treat the degenerate IRs $(0,0,c)$ as an example.

2. Generating function for branching rules

When polynomial basis states are required for a group–subgroup pair one should start by calculating the relevant branching rules generating function; generating function techniques are explained in a forthcoming review article (Gaskell *et al* 1992; see also Gaskell *et al* 1978 for an early version). For $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$ the branching rules generating function is a rational function F in six dummy variables A, B, C, S, T

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and U which carry as exponents the three representation labels of SO(7) and those of the three SU(2) subgroups respectively. The power series expansion of F

$$F = \sum_{\substack{abc \\ stu}} A^a B^b C^c S^s T^t U^u c_{abc,stu} \tag{2.1}$$

gives, as the coefficient of $A^a B^b C^c S^s T^t U^u$, the multiplicity of the $\mathrm{SU}(2)^3$ irreducible representation (IR) (s,t,u) in the SO(7) IR (a,b,c).

To calculate the $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$ generating function from first principles, one might start with the SO(7) character generator. But it contains 127 terms, each a fraction with 12 denominator factors and a polynomial numerator (Gaskell 1983; see also Baclawski 1983 and King and El-Sharkaway 1984); we deem this approach impracticable.

Fortunately, two other approaches are available. The first uses the subgroup chain $\mathrm{SO}(7) \supset \mathrm{SO}(6) \sim \mathrm{SU}(4) \supset \mathrm{SU}(2)^2 \times \mathrm{U}(1)$; the SU(2) subgroups are two of the three we want, labelled by s and t in equation (2.1), while U(1) labels refer to the weights of the third, labelled by u in (2.1); it is known how to convert a generating function for weights into the corresponding one for IRs (Gaskell *et al* 1978, Gaskell *et al* 1992 or see below).

The second alternative approach, which we adopt, makes use of the group–subgroup pair $\mathrm{SO}(7) \supset \mathrm{SO}(5) \times \mathrm{U}(1)$, whose generating function is given by Patera *et al* (1980). Then we convert U(1) whose labels are SU(2) u weights into the corresponding IRs to obtain the (new) branching rule generating function for the subjoining $\mathrm{SO}(7) > \mathrm{SO}(5) \times \mathrm{SU}(2)$ (alternatively use the known generating functions for $\mathrm{SO}(7) > \mathrm{Sp}(6)$ (Patera *et al* 1980) and $\mathrm{Sp}(6) \supset \{\mathrm{Sp}(4) \sim \mathrm{SO}(5)\} \times \mathrm{SU}(2)$). Finally we use the known $\mathrm{SO}(5) \supset \mathrm{SU}(2)^2$ generating function to obtain the desired generating function for $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$.

Our starting point is then the $\mathrm{SO}(7) \supset \mathrm{SO}(5) \times \mathrm{U}(1)$ branching rules generating function

$$\begin{aligned} &\frac{1}{(1-AZ^2)(1-AZ^{-2})(1-BEZ^2)(1-BEZ^{-2})(1-CDZ)(1-CDZ^{-1})} \\ &\qquad \times \left[\frac{1}{(1-AE)(1-C^2E)} + \frac{B}{(1-C^2E)(1-B)} + \frac{BD^2}{(1-B)(1-BD^2)} \right. \\ &\qquad \left. + \frac{ABD^2E}{(1-BD^2)(1-AE)} \right] \end{aligned} \tag{2.2}$$

where the dummy variables A, B and C carry the SO(7) Dynkin labels, D and E carry SO(5) labels and Z carries the U(1) label.

To convert U(1) to SU(2) (u), it is necessary to multiply by $(1-Z)(1-UZ)^{-1}$ and retain the constant term in Z. The result is

$$\begin{aligned} &\frac{1}{(1+A)(1+BE)(1-A^2C^2D^2)(1-BEU^2)(1-ABE)(1-CDU)} \\ &\qquad \times \left[\frac{1+BCDEU+ABCDEU+ABC^2D^2E}{(1-BC^2D^2E)} \right. \\ &\qquad \left. + \frac{AU^2+A^2BCDEU^3+ABEU^2+ACDU}{(1-AU^2)} \right] \end{aligned} \tag{2.3}$$

multiplied by the factor in the square bracket of (2.2) above. It is the branching rule generating function for $SO(7) > SO(5) \times SU(2)$, with U carrying the $SU(2)$ label u. Putting everything over a common denominator we fortunately get rid of the troublesome denominator factors $1 - ABE$ and $1 - BC^2D^2E$. The result is

$$\begin{aligned}&[(1+A)(1+BE)(1-AC^2D^2)(1-BEU^2)(1-CDU)(1-AU^2)(1-AE)\\&\quad\times(1-BD^2)(1-B)(1-C^2E)]^{-1}[1+BCDEU+ABCDEU\\&\quad+ABC^2D^2E+ABEU^2+ACDU-ABCDEU^3-A^2BC^2D^2EU^2\\&\quad-ABC^2D^2EU^2-A^2B^2C^3D^3E^2U^3-AB^2C^2D^2E^2U^2\\&\quad-ABC^3D^3EU].\end{aligned} \tag{2.4}$$

The remaining task is to eliminate the group $SO(5)$ in favour of its subgroup $SU(2)^2$; the $SO(5) \supset SU(2)^2$ branching rule generating function is known to be

$$[(1-DS)(1-DT)(1-E)(1-EST)]^{-1} \tag{2.5}$$

where ST carry the IR labels of the two $SU(2)$ subgroups. To eliminate $SO(5)$, multiply (2.5), with D and E replaced by their reciprocals, by (2.4) and retain the coefficient of D^0E^0. We carried this out with the help of MAPLE, but the result (everything over a common denominator) is not immediately in a desirable form; it contains 14 denominator factors and a numerator consisting of 420 terms, of both signs. It is interesting that the numerator merely changes sign under the replacements $A^a \to A^{4-a}$, $B^b \to B^{4-b}$, $C^c \to C^{8-c}$, $S^s \to S^{6-s}$, $T^t \to T^{6-t}$ and $U^u \to U^{4-u}$. A desirable form has everything positive (i.e. no cancellations), so that we can interpret it in terms of an integrity basis (elementary multiplets) and construct polynomial basis states.

For the $G \supset H$ branching rule generating function the number of denominator factors in each term should be $\frac{1}{2}(r_G + l_G)$, the number of G labels, internal and external, less the number of internal H labels, $\frac{1}{2}(r_H - l_H)$ (Racah 1965, p 57); r_G, r_H, l_G and l_H are the order (r) and rank (l) of the group G and the subgroup H respectively; hence the $SO(7) \supset SU(2)^3$ generating function requires 9 denominator factors ($r_{SO(7)} = 21$, $l_{SO(7)} = 3$, $r_{SU(2)^3} = 9$ and $l_{SU(2)^3} = 3$).

To put our generating function in desirable form the numerator must be written as a sum of terms, each containing a product of five denominator factors and a positive coefficient. We know of no algorithm for such a separation.

To make some headway, we first looked at the degenerate cases (one or more $SO(7)$ labels zero). For $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, $(a,b,0)$, $(a,0,c)$ and $(0,b,c)$ the number of denominators factors are 3,4,5,7,7,7 respectively (for a generalization of Racah's formula for the number of internal labels in a degenerate case, see Seligman and Sharp 1983). The generating functions are easily found in a desirable form. For the cases with two labels non-zero we find

$$(ab0): \qquad \frac{1}{abhh'ik}\left[\frac{1+j}{g} + \frac{c+n+p+np}{c}\right] \tag{2.6}$$

$$(a0c): \qquad \frac{1}{acdd'f}\left[\frac{1+l}{bm} + \frac{m'+l'}{bm'} + \frac{e}{em} + \frac{em'}{em'}\right] \tag{2.7}$$

$$(0bc): \qquad \frac{1}{dd'hh'k}\left[\frac{1+j}{gi} + \frac{f+q+q'+qq'}{fi} + \frac{e+r}{fe}\right]. \tag{2.8}$$

For brevity we have adopted the convention that in a denominator factor a letter stands for unity minus that letter. The letters (elementary multiplets) are defined as follows:

$$
\begin{array}{llll}
a = AST & b = AU^2 & c = A^2 & d = CSU \\
d' = CTU & e = C^2ST & f = C^2 & g = BSTU^2 \\
h = BS^2 & h' = BT^2 & i = BU^2 & j = B^2STU^2 \\
k = B^2 & l = ACSU & l' = ACTU & m = AC^2S^2 \\
m' = AC^2T^2 & n = ABST & p = ABU^2 & q = BCSU \\
q' = BCTU & r = BC^2ST & s = ABCSU & s' = ABCTU \\
t = ABC^2S^2 & t' = ABC^2T^2 & u = ABC^2STU^2. &
\end{array}
\tag{2.9}
$$

The last five, which appear only in the general generating function, equation (2.10) below, are included here for convenience.

There are five sets of ten elementary multiplets in each of which the elementary multiplets are compatible in pairs (and others with nine elementary multiplets—the required number). Each set includes $acdd'fhh'k$; the other two are respectively bi, bm, bm', em, and em'. The first set is evident from equations (2.6)–(2.8). For the compatibility of h,h' and k with m and m' in the other four sets it is necessary to look also at the branching rule for the SO(7) IRs (112) and (122). With the plausible assumption that elementary multiplets compatible in pairs are compatible in any combination we have five terms in the general generating function each with ten denominator factors, in violation of Racah's counting, according to which there must be no more than nine. The cure for the disease is to make appropriate triples incompatible even though each pair in them is compatible. It seems that afi, bmh', aek and $bm'h$ are the only possible choices; each has all three SO(7) labels non-zero, so the change does not affect $(ab0)$, $(a0c)$ or $(0bc)$ and each appears with negative sign in the numerator of the generating function with everything over a common denominator. This leads to the generating function below, equation (2.10) (at least its denominator factors). It remains to find a choice of numerator terms that will that will reproduce the generating function calculated from (2.4) and (2.5). The degenerate cases (2.6)–(2.8) provide useful information. This can be augmented using the Grobner basis package in MAPLE (see Char *et al* (1988), Buchberger (1987)) to reduce the numerator modulo carefully chosen ideals generated by certain denominator terms which eliminate all but one set of numerator terms. Although this is not an algorithmic process it turns out that the information obtained is sufficient to find a possible form of the generating function without too much difficulty. As an example of this process let $I_1 = (1 - BU^2, 1 - BSTU^2)$, the ideal generated in $R = Q[A, B, C, S, T, U]$ by the denominator terms $1 - BU^2$ and $1 - BSTU^2$. We find $I_1 = (BU^2 - 1, ST - 1)$ with $BU^2 - 1$ and $ST - 1$ being a Grobner basis with respect to, for example, the deglex term ordering. Considering (2.10) we can see that when reduced modulo I_1 the common numerator is equivalent to $(1-A^2)(1-C^2)^2(1-AC^2S^2)(1-AC^2T^2)(g+j)$, but this factorization may not be apparent since equality is only modulo terms in I_1. To remove the extra factors we can work, for example, modulo $I_2 = (BU^2 - 1, ST - 1, A^{12}, C^{12})$, since the extra factors have inverses in R/I_2 ($(1-A^2)^{-1} = 1 + \sum_{i=1}^{5} A^{2i}$ etc). The result is that we can find the numerator of the sixth term in (2.10) modulo I_2, it is $1+b$. Now the rewrite rules for I_2 are $BU^2 \mapsto 1$, $ST \mapsto 1$, $A^{12} \mapsto 0$ and $C^{12} \mapsto 0$. Assuming that there are no numerator terms involving 12th powers of A or C these rules preserve the number of numerator terms. We conclude that the sixth term in (2.10) must have exactly two numerator terms. Comparing with (2.8) suggests that the second term is the reduction of j. We chose the second to be g, but i and 1 would also be consistent. Continuing in this way, removing terms as they are produced, leads to (2.10).

The final result is

$$\begin{aligned}\frac{1}{dd'}\Bigg[&\frac{1+aq+aq'+afn+nq+nq'+u+nu}{abcfhh'k}\\ &+\frac{fi+q+q'+qq'+fp+fn+pq+pq'}{bcfhh'ik}+\frac{m+l+s+bt}{abcfhkm}\\ &+\frac{m'+l'+s'+bt'}{abcfh'km'}+\frac{i+n+p+np}{abchh'ik}+\frac{g+j}{abghh'ik}\\ &+\frac{h'm+t}{acfhh'km}+\frac{hm'+t'}{acfhh'km'}+\frac{ek+r}{cefhh'km}+\frac{ekm'+m'r}{cefhh'km'}\\ &+\frac{e}{acefhh'm}+\frac{em'}{acefhh'm'}\Bigg].\end{aligned} \tag{2.10}$$

Again we have used the convention that in a denominator factor a letter stand for unity minus that letter. The letters were defined in equation (2.9).

3. Polynomial basis states

For the construction of polynomial basis states we follow an approach suggested by de Guise and Sharp (1991) in connection with the $SU(3) \supset DT$ and $SO(5) \supset DT$ problems. This section's discussion is general, using $SO(7) \supset SU(2)^3$ only as an example. We will speak of a simple compact Lie group G, of rank l, and when required, a subgroup H.

For the IR $(a_1, \ldots, a_l)$ our basis states are to be polynomials of degree a_i in the basis states of the ith fundamental IR of G. That means that only stretched (IR labels additive) IRs in the direct product of a_i copies of each fundamental IR are to be retained. Looking only at parts of total degree two we have for $SO(7)$

$$(1,0,0)^2_{28} \supset (2,0,0)_{27} + (0,0,0)_1 \tag{3.1a}$$
$$(0,1,0)^2_{231} \supset (0,2,0)_{168} + (2,0,0)_{27} + (0,0,2)_{35} + (0,0,0)_1 \tag{3.1b}$$
$$(0,0,1)^2_{36} \supset (0,0,2)_{35} + (0,0,0)_1 \tag{3.1c}$$
$$(1,0,0) \times (0,1,0)_{147} \supset (1,1,0)_{105} + (0,0,2)_{35} + (1,0,0)_7 \tag{3.1d}$$
$$(1,0,0) \times (0,0,1)_{56} \supset (1,0,1)_{48} + (0,0,1)_8 \tag{3.1e}$$
$$(0,1,0) \times (0,0,1)_{168} \supset (0,1,1)_{112} + (1,0,1)_{48} + (0,0,1)_8. \tag{3.1f}$$

The square of an IR above means the symmetric (polynomial) part of the direct product of two copies. A subscript on an IR or product is its dimension. The stretched part of each product is the first IR on the right. The states of all the other IRs are unwanted for the purpose of forming our polynomial basis (we call them elementary unwanted states). There may be more elementary unwanted states of higher degree; their number, however is finite (by the Hilbert basis theorem).

To deal consistently with the unwanted states we need the help of the group's character generator in a positive form. Gaskell (1983) has given a general algorithm for its construction. We may characterize it by listing the incompatible pairs (or triples etc, with all members of each subset mutually compatible) of fundamental states. 'Incompatible' means that the states never appear multiplied for the purposes of forming states of higher IRs; a state is incompatible with itself if its square never appears. Different versions of the character generator are possible (they are identical when put over a common denominator). Any valid version yields a consistent set of incompatibilities.

Each incompatible pair (triple etc) is one term in the expression for an elementary unwanted state. We formally set each elementary unwanted state equal to zero (i.e. work modulo the ideal they generate) and eliminate each incompatible pair (triple etc) in favour of the other terms whenever it arises. The surviving states of degree a_i in the ith fundamental basis states constitute a complete, non-redundant basis for the IR (a_i).

However, they generally contain admixtures of IRs lower than their degree would indicate. Sometimes (not usually) one can define 'traceless' variables corresponding to the basis states of certain fundamental IRs such that products of powers of them contain no unwanted states (Lohe and Hurst 1971, Patera *et al* 1989, section 4 of this paper). The same effect is obtained by operating with P, an instruction to retain only the wanted part of its operand. Since P commutes with group generators or functions of them such as a Hamiltonian lying in the enveloping algebra, one can ignore the role of P and work with the states as they stand when calculating generator matrix or matrix elements of, for example, a Hamiltonian or missing label generator (a subalgebra scalar in the group's enveloping algebra). The matrix element $(i \mid \Omega \mid j)$ is the coefficient of the basis state $|\, i)$ when the operator Ω acts on the state $|\, j\rangle$; the round bracket notation is used to indicate that the matrix element is not defined by a scalar product.

Our states at this point do not recognize any subgroup (except $\mathrm{U}(1)^l$ corresponding to the weight components). To organize them into multiplets of a subgroup H we need the $G \supset H$ branching rules generating function. It defines elementary multiplets. Construct their highest weight states as polynomials in the fundamental basis states, eliminating incompatible pairs, triples etc, as explained in the second preceding paragraph. The highest state of an arbitrary subgroup multiplet is given by the appropriate product of powers of highest states of elementary multiplets. Lower states are obtained by operating with lowering subalgebra generators, eliminating incompatible combinations of fundamental basis states wherever they arise. The basis states are not orthogonal in general (except when the subgroup provides enough labels to specify them completely). For the purpose of calculating matrix elements of generators etc, there is no need to orthonormalize them.

4. The degenerate $\mathrm{SO}(7)$ irreducible representations $(0,0,c)$

In this section we construct the $\mathrm{SO}(7) \supset \mathrm{SU}(2)^3$ basis states for the degenerate representations $(0,0,c)$. There are no missing labels so the states may be normalized explicitly; we then evaluate the generator matrix elements. Apart from those of the subalgebra $SU(2)^3$ the generators comprise an $SU(2)^3$ tensor $G^{(1,1,2)}$. We suppress the representation labels $(1,1,2)$ and exhibit the three component labels as subscripts where necessary; G has 12 components, G_{ijk} where $i=\pm1$, $j=\pm1$ and $k=\pm2,0$.

We name the eight basis states of $(0,0,1)$ as

$$\alpha=[1,0,1] \qquad \beta=[0,1,1] \qquad \gamma=[0,\bar{1},1]$$
$$\delta=[\bar{1},0,1] \qquad \alpha^*=[\bar{1},0,\bar{1}] \qquad \beta^*=[0,\bar{1},\bar{1}]$$
$$\gamma^*=[0,1,\bar{1}] \qquad \delta^*=[1,0,\bar{1}].$$

The numbers in the square brackets are the $SU(2)^3$ weight components $[m_s,m_t,m_u]$; in a fundamental weights basis the components are $\lambda_1=(m_u-m_s-m_t)/2$, $\lambda_2=m_t$, $\lambda_3=m_s-m_t$.

The subalgebra root generators may be taken as the differential operators

$$\begin{aligned}
S_+ &= \alpha\partial_\delta+\delta^*\partial_{\alpha^*} & S_- &= \delta\partial_\alpha+\alpha^*\partial_{\delta^*}\\
T_+ &= \beta\partial_\gamma+\gamma^*\partial\beta^* & T_- &= \gamma\partial_\beta+\beta^*\partial_{\gamma^*}\\
U_+ &= \alpha\partial_{\delta_*}+\delta\partial\alpha^*-\beta\partial_{\gamma^*}-\gamma\partial_{\beta^*} & U_- &= \delta^*\partial_\alpha+\alpha^*\partial_\delta-\gamma^*\partial_\beta-\beta^*\partial_\gamma .
\end{aligned} \tag{4.1}$$

We give one component of G

$$G_{\bar{1},\bar{1},2}=\gamma\partial_{\delta^*}+\delta\partial_{\gamma^*}. \tag{4.2}$$

The other 11 components may be obtained from (4.2) by applying the $SU(2)^3$ root generators. The generators corresponding to the simple roots $\alpha_1,\alpha_2,\alpha_3$ of $SO(7)$ are respectively $G_{\bar{1},\bar{1},2}$, T_+ and $G_{1,\bar{1},0}$.

The character generator for $(0,0,c)$ IRs according to Gaskell (1983) may be written as

$$\frac{1}{(1-\alpha)(1-\alpha^*)(1-\beta)(1-\beta^*)(1-\gamma)(1-\gamma^*)}\left[\frac{1}{(1-\delta)}+\frac{\delta^*}{(1-\delta^*)}\right]. \tag{4.3}$$

According to (4.3) δ and δ^* are incompatible. The reason is an unwanted scalar, the second term on the right hand side of equation (3.1c). It is $M=\alpha\alpha^*-\beta\beta^*+\gamma\gamma^*-\delta\delta^*$; we set $M=0$ and eliminate $\delta\delta^*$ whenever it arises by writing

$$\delta\delta^*=\alpha\alpha^*-\beta\beta^*+\gamma\gamma^*. \tag{4.4}$$

The $SO(7)\supset SU(2)^3$ branching rules generating function for $(0,0,c)$ is obtained from (2.7) by setting $A=0$, from (2.8) by setting $B=0$ or from (2.10) by setting $A=B=0$. It is

$$\frac{1}{(1-d)(1-d')(1-e)(1-f)}. \tag{4.5}$$

The elementary multiplets d,d',e and f are defined in equation (2.9). Explicitly we find

$$d=\alpha \qquad d'=\beta \qquad e=\alpha\gamma^*+\beta\delta^* \qquad f=\alpha\alpha^*+\beta\beta^*-\gamma\gamma^*-\delta\delta^*. \tag{4.6}$$

We made f specific by requiring that it be orthogonal to the scalar M above. We must eliminate the incompatible product $\delta\delta^*$ by means of (4.4) above. Dropping a factor of two we get

$$f = \beta\beta^* - \gamma\gamma^*. \tag{4.6a}$$

Do not forget that the operator P is understood to operate on all our states from now on. It projects out unwanted parts.

Thus we find for the highest state of the general $\mathrm{SU}(2)^3$ multiplet

$$\left|\begin{matrix} 0 & 0 & c \\ s & t & u \\ s & t & u \end{matrix}\right\rangle = N_{stu}\alpha^a\beta^b(\alpha\gamma^* + \beta\delta^*)^d(\beta\beta^* - \gamma\gamma^*)^e. \tag{4.7}$$

Here

$$a = \tfrac{1}{2}(s+u-t) \qquad b = \tfrac{1}{2}(t+u-s) \qquad d = \tfrac{1}{2}(s+t-u) \qquad e = \tfrac{1}{2}(c-s-t) \tag{4.8}$$

or

$$s = a+d \qquad t = b+d \qquad u = a+b \qquad c = a+b+2d+2e \tag{4.9}$$

N_{stu} is a normalization factor, to be determined below. Since the exponents in (4.7) are non-negative integers, we have the branching rules

$$c \geqslant s+t \geqslant u \geqslant |s-t| \tag{4.10}$$

with $s+t$ and u having the parity of c. We mention in passing two methods of extracting the wanted part of a state. The first is to replace the variables $\alpha, \beta, \gamma, \delta, \alpha^*, \beta^*, \gamma^*$ and δ^* by their 'traceless' analogues

$$\alpha' = \alpha - M(N+4)^{-1}\partial_{\alpha^*} \qquad \beta' = \beta + M(N+4)^{-1}\partial_{\beta^*} \tag{4.11}$$

etc. The sign of the second term in each of (4.11) is negative for $\alpha', \gamma', \alpha^{*\prime}, \gamma^{*\prime}$ and positive for each of $\beta', \beta^{*\prime}, \delta', \delta^{*\prime}$. Here M is the unwanted scalar under discussion and N is the degree operator.

Another method is to write (the projection operator P is understood not to be operating in (4.12) except where shown explicitly)

$$P\alpha^a\beta^b(\alpha\gamma^*+\beta\delta^*)^d(\beta\beta^*-\gamma\gamma^*)^e = \sum_{x=0}^{e} A_x\alpha^a\beta^b(\alpha\gamma^*+\beta\delta^*)^d(\beta\beta^*-\gamma\gamma^*)^{e-x}M^x. \tag{4.12}$$

Operate on both sides with $M^\dagger = \partial_\alpha\partial_{\alpha^*} - \partial_\beta\partial_{\beta^*} + \partial_\gamma\partial_{\gamma^*} - \partial_\delta\partial_{\delta^*}$. The left hand side then vanishes (it was orthogonal to anything containing M as a factor) and we get a recursion formula for A_x whose solution is

$$A_x = \frac{e!(b+d+e+1)!(a+b+2d+2e-x+2)!}{(a+b+2d+2e+2)!x!(e-x)!(b+d+e-x+1)!}. \tag{4.13}$$

We now calculate generator matrix elements with respect to the basis states (4.7) and incidentally determine the normalization constant N_{stu}. The subalgebra matrix elements are trivial. The matrix elements of G are given by the Wigner–Eckart theorem in terms of the reduced matrix elements by

$$\left\langle \begin{matrix} 0 & 0 & c \\ s' & t' & u' \\ m'_s & m'_t & m'_u \end{matrix} \middle| G_{ijk} \middle| \begin{matrix} 0 & 0 & c \\ s & t & u \\ m_s & m_t & m_u \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} 0 & 0 & c \\ s' & t' & u' \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2}s & \frac{1}{2} \\ \frac{1}{2}m_s & \frac{1}{2}i \end{matrix} \middle| \begin{matrix} \frac{1}{2}s' \\ \frac{1}{2}m'_s \end{matrix} \right\rangle$$
$$\times \left\langle \begin{matrix} \frac{1}{2}t & \frac{1}{2} \\ \frac{1}{2}m_t & \frac{1}{2}j \end{matrix} \middle| \begin{matrix} \frac{1}{2}t' \\ \frac{1}{2}m'_t \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2}u & 1 \\ \frac{1}{2}m_u & \frac{1}{2}k \end{matrix} \middle| \begin{matrix} \frac{1}{2}u' \\ \frac{1}{2}m'_u \end{matrix} \right\rangle$$
$$\times \left((s'+1)(t'+1)(u'+1)\right)^{-1/2}. \tag{4.14}$$

It is necessary for us to give only the reduced matrix elements of G.

We start by writing

$$G_{\bar{1},\bar{1},\bar{2}} \left| \begin{matrix} 0 & 0 & c \\ s & t & u \\ s & t & u \end{matrix} \right\rangle = \sum_{ijk} A_{ijk} \left| \begin{matrix} 0 & 0 & c \\ s+i & t+j & u+k \\ s-1 & t-1 & u-2 \end{matrix} \right\rangle \tag{4.15}$$

where i and j take the values 1 and -1 and k takes the values $2, 0$ and -2. Operate on both sides with $S_+ T_+ U_+^2$. All the terms on the right except that with $i = j = 1$, $k = 2$ are annihilated and we find

$$A_{112} = -\frac{(c-s-t)}{2(s+1)(t+1)(u+1)(u+2)} \frac{N_{stu}}{N_{s+1,t+1,u+2}}. \tag{4.16}$$

We now calculate the A_{ijk} one by one by transferring to the left hand side of (4.15) the terms involving A already determined and applying raising subalgebra generators to annihilate all but one term remaining on the right. The last A to be determined is (using MAPLE)

$$A_{\bar{1},\bar{1},\bar{2}} = \frac{(s+u-t)(t+u-s)(s+t+u)(s+t+u+2)(c+s+t+4)}{32(s+1)(t+1)u(u+1)}$$
$$\times \frac{N_{s,t,u}}{N_{s-1,t-1,u-2}}. \tag{4.17}$$

Because $G_{\bar{1},\bar{1},\bar{2}} = -(G_{112})^\dagger$ we have the equality (the two sides are matrix elements of Hermitian conjugate operators between the same states in reverse order

$$A_{\bar{1},\bar{1},\bar{2}}(s+1,t+1,u+2) = -\left((s+1)(t+1)(u+1)(u+2)/2\right)^{1/2} A_{112}(s,t,u). \tag{4.18}$$

This implies the recurrence formula for the normalization constants

$$\frac{N_{s+1,t+1,u+2}}{N_{s,t,u}} = \left\{16(c-s-t)(s+2)(t+2)(u+2)(u+3) \times \left[(s+u-t+2)\right.\right.$$
$$\left.\left. \times (t+u-s+2)(s+t+u+4)(s+t+u+6)(c+s+t+6)\right]^{-1}\right\}^{1/2}. \tag{4.19}$$

Iterating (4.19) gives

$$N_{s,t,u} = \left[\frac{(s+1)!(t+1)!(u+1)!}{\{\frac{1}{2}(c-s-t)\}!\{\frac{1}{2}(s+u-t)\}!\{\frac{1}{2}(t+u-s)\}!}\right]^{1/2} \times \left[\{\tfrac{1}{2}(s+t+u+2)\}!\{\tfrac{1}{2}(c+s+t+4)\}!\right]^{1/2}\kappa \tag{4.20}$$

where κ is constant along $s = s_0 - n$, $t = t_0 - n$, $u = u_0 - 2n$. Choose $s_0 + t_0 = c$, then $n = \frac{1}{2}(c-s-t)$, $s_0 = \frac{1}{2}(c+s-t)$, $t_0 = \frac{1}{2}(c-s+t)$, $u_0 = c+u-s-t$.

(s_0, t_0, u_0) is at the boundary of the $(0,0,c)$ weight diagram where there are no unwanted states and we can use simpler methods (not involving generators) to evaluate κ of (4.20). Our state is

$$\left|\begin{matrix} 0 & 0 & c \\ s_0 & t_0 & u_0 \\ s_0 & t_0 & u_0 \end{matrix}\right\rangle = N_{s_0,t_0,u_0}\alpha^{(c+u-2t)/2}\beta^{(c+u-2s)/2}(\alpha\gamma^* + \beta\delta^*)^{(s+t-u)/2}$$
$$= N_d \alpha^a\beta^b(\alpha\gamma^* + \beta\delta^*)^d = |d\rangle \tag{4.21}$$

say. Operating with the Hermitian conjugate operators $\alpha\gamma^* + \beta\delta^*$ and $\partial_\alpha\partial_{\gamma^*} + \partial_\beta\partial_{\delta^*}$ between the states $|d\rangle$ and $|d+1\rangle$ gives the recursion relation

$$\frac{N_{d+1}}{N_d} = \left((a+1)(a+b+d+2)\right)^{-1/2} \tag{4.22}$$

whose solution is

$$N_d = N_{s_0,t_0,u_0} = \left(\frac{(a+b+1)!}{a!b!d!(a+b+d+1)!}\right)^{1/2}. \tag{4.23}$$

This determines κ of (4.20) to be

$$\kappa = \left[\frac{(c+2)!}{\{\frac{1}{2}(c+s-t+2)\}!\{\frac{1}{2}(c-s+t+2)\}!\{\frac{1}{2}(s+t-u)\}!}\right]^{1/2} \tag{4.24}$$

and hence

$$N_{s,t,u} = [(c+2)!(s+1)!(t+1)!(u+1)!]^{1/2} \times \left[\{\tfrac{1}{2}(c-s-t)\}!\{\tfrac{1}{2}(s+u-t)\}! \times \{\tfrac{1}{2}(t+u-s)\}!\{\tfrac{1}{2}(s+t+u+2)\}!\{\tfrac{1}{2}(c+s+t+4)\}! \times \{\tfrac{1}{2}(c+s-t+2)\}!\{\tfrac{1}{2}(c-s+t+2)\}!\{\tfrac{1}{2}(s+t-u)\}!\right]^{-1/2}. \tag{4.25}$$

It is remarkable that the wanted part of the state $\alpha^a\beta^b(\alpha\delta^* + \beta\delta^*)^d(\beta\beta^* - \gamma\gamma^*)^e$ has been normalized without it ever being isolated.

Since we know the generator matrix elements A_{ijk} in equation (4.15), it is straightforward to find the reduced matrix elements. Because $G_{-i,-j,-k} = (-1)^{(i-j+k)/2}G_{ijk}$ it follows that

$$\left\langle\begin{matrix} 0 & 0 & c \\ s+i & t+j & u+k \end{matrix}\right\| G \left\|\begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix}\right\rangle = (-1)^{(i-j+k)/2}\left\langle\begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix}\right\| G \left\|\begin{matrix} 0 & 0 & c \\ s+i & t+j & u+k \end{matrix}\right\rangle. \tag{4.26}$$

Hence we need to give only half the reduced matrix elements, say those for which $i=-1$ in (4.26). The results are

$$\left\langle \begin{matrix} 0 & 0 & c \\ s-1 & t-1 & u-2 \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle$$
$$= \tfrac{1}{8}\big[(s+u-t)(t+u-s)(s+t+u)(s+t+u+2)(c+s+t+4)$$
$$\times (c-s-t+2)/u\big]^{1/2}$$

$$\left\langle \begin{matrix} 0 & 0 & c \\ s-1 & t+1 & u-2 \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle$$
$$= \tfrac{1}{8}\big[(s+u-t)(s+u-t-2)(s+t+u+2)(c+s-t+2)$$
$$\times (c-s+t+4)(s+t-u+2)/u\big]^{1/2}$$

$$\left\langle \begin{matrix} 0 & 0 & c \\ s-1 & t-1 & u+2 \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle$$
$$= \tfrac{1}{8}\big[(s-u+t)(s-u+t-2)(c+s+t+4)(c-s-t+2)$$
$$\times (s+u-t+2)(t+u-s+2)/(u+2)\big]^{1/2}$$

$$\left\langle \begin{matrix} 0 & 0 & c \\ s-1 & t+1 & u+2 \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle$$
$$= \tfrac{1}{8}\big[(s+t-u)(t+u-s+2)(t+u-s+4)(s+t+u+4)$$
$$\times (c+s-t+2)(c+t-s+4)/(u+2)\big]^{1/2}$$

$$\left\langle \begin{matrix} 0 & 0 & c \\ s-1 & t-1 & u \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle$$
$$= \tfrac{1}{4}(s-t)\big[(s+t+u+2)(s+t-u)(c+s+t+4)(c-s-t+2)$$
$$\times (u+1)/2u(u+2)\big]^{1/2}$$

$$\left\langle \begin{matrix} 0 & 0 & c \\ s-1 & t+1 & u \end{matrix} \middle\| G \middle\| \begin{matrix} 0 & 0 & c \\ s & t & u \end{matrix} \right\rangle$$
$$= -\tfrac{1}{4}(s+t+2)\big[(s+u-t)(t+u-s+2)(c+s-t+2)$$
$$\times (c-s+t+4)(u+1)/2u(u+2)\big]^{1/2}. \qquad (4.27)$$

Our generator matrix elements are in agreement with those given by Van der Jeugt and De Wilde (1984). Their paper, however, gives only the magnitude squared of the reduced matrix elements (no phases) and there is a typographical error in their equation (4.26); u should be 1 in the last factor.

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